

# LIMIT DISTRIBUTIONS OF CURVES IN $SO(n, 1)$ ACTING ON HOMOGENEOUS SPACES UNDER GEODESIC FLOWS

Lei Yang\*

## Abstract

Let  $G$  be a Lie group and  $\Gamma$  be a lattice of  $G$ , i.e.,  $G/\Gamma$  admits a finite  $G$ -invariant measure. We consider the actions of  $SO(n, 1)$  on  $G/\Gamma$ , in other words, we assume  $SO(n, 1) \subseteq G$ , and  $SO(n, 1)$  acts on  $G/\Gamma$  by left multiplication. Further, we assume that  $\overline{SO(n, 1)g\Gamma} = G$  for some  $g \in G$ . Then for any compact segment of analytic curve  $\phi : I \rightarrow SO(n, 1)$  and  $x = g\Gamma \in G/\Gamma$ ,  $\phi(I)x$  gives a curve in the space. In this article, we consider the limit distributions of such curves under geodesic flow.

Since  $SO(n, 1)/SO(n - 1)$  can be identified as the unit tangent bundle of the universal hyperbolic  $n$ -space, say  $T^1(\mathbb{H}^n)$ , and the geodesic flow corresponds to the  $A$ -action, where  $A$  is a maximal connected  $\mathbb{R}$ -diagonalizable subgroup, since  $SO(n, 1)$  is of  $\mathbb{R}$ -rank one,  $A$  is a one-parameter group. And there is a visual map sending every point in  $T^1(\mathbb{H}^n)$  to the ideal boundary sphere  $\partial\mathbb{H}^n$ . It is shown that if the visual map doesn't send the curve into a proper subsphere of  $\partial\mathbb{H}^n$ , then under geodesic flow the orbit  $\phi(I)x$  gets asymptotically equidistributed on  $G/\Gamma$ . This problem was proposed by Nimish Shah in [3] and is a generalization of the main result in that paper. The proof borrows the main technique from that paper but needs some new observations on the representations of  $SL(2, \mathbb{R})$ .

## 1 Introduction

We consider a Lie group  $G$  and a discrete subgroup  $\Gamma$  such that  $G/\Gamma$  admits a finite  $G$ -invariant measure. Let  $\pi : G \rightarrow G/\Gamma$  be the canonical projection. Assume  $H = SO(n, 1)$  is a subgroup of  $G$  and one orbit of  $H$  on  $G/\Gamma$  is dense, i.e.,  $\overline{Hg\Gamma} = G$  for some  $g \in G$ , we shall fix this  $g$  throughout this article.

Remark: The assumption that  $Hg\Gamma$  is dense in  $G$  is not special for the following reason: since  $SO(n, 1)$  is generated by its one parameter unipotent subgroups, then by the topological version of

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\*Department of Mathematics, The Ohio State University, email address: [lyang@math.osu.edu](mailto:lyang@math.osu.edu)

Ratner's theorem (see [9]), the closure of  $Hg\Gamma$  is homogeneous, i.e., there exists some closed subgroup  $F$  of  $G$  containing  $H$  such that  $F \cap g\Gamma g^{-1}$  is a lattice of  $F$  and moreover  $\overline{Hg\Gamma} = Fg\Gamma$ . This implies that  $Hg\Gamma g^{-1}$  is dense in  $Fg\Gamma g^{-1}$ . Since  $Fg\Gamma g^{-1} \cong F/F \cap g\Gamma g^{-1}$ , we will have  $H(F \cap g\Gamma g^{-1})$  is dense in  $F/F \cap g\Gamma g^{-1}$ . Therefore we can replace  $G$  by  $F$ ,  $\Gamma$  by  $F \cap g\Gamma g^{-1}$  and  $g$  by  $e$  to make the same condition hold.

We identify  $H/\mathrm{SO}(n-1)$  as the unit bundle of  $n$ -dimensional hyperbolic space  $T^1(\mathbb{H}^n)$ . Let  $\tau : \mathrm{SO}(n, 1) \rightarrow T^1(\mathbb{H}^n)$  denote the canonical projection.

The visual map

$$\mathrm{Vis} : T^1(\mathbb{H}^n) \rightarrow \partial\mathbb{H}^n \cong \mathbb{S}^{n-1} \quad (1)$$

is defined by sending every unit vector  $v \in T^1(\mathbb{H}^n)$  to the ideal boundary along the geodesic flow. We denote the geodesic flow by  $\{g_t\}$ . We consider a compact analytic curve  $\phi : I = [a, b] \rightarrow H$ , we will prove the following theorem:

**Theorem 1.1.** *Let  $G/\Gamma$ ,  $H$  and  $\phi$  be as above. Then if the visual map doesn't send  $\tau(\phi(I))$  into a proper subsphere of  $\partial\mathbb{H}^n$ , then the orbit of the curve acting on  $x = \pi(g)$  will get asymptotically equidistributed under the geodesic flow. In other words, for any  $f \in C_c(G/\Gamma)$  and any  $x \in G/\Gamma$  we have:*

$$\lim_{t \rightarrow \infty} \frac{1}{|I|} \int_I f(g_t \phi(s)x) ds = \int_{G/\Gamma} f d\mu_G \quad (2)$$

where  $\mu_G$  denotes the probability  $G$ -invariant measure.

In [3], Nimish Shah proved the theorem for  $G = \mathrm{SO}(m, 1)$  for some  $m \geq n$ , but for general case when  $G$  is an arbitrary Lie group containing  $\mathrm{SO}(n, 1)$ , that argument does not work since it depends on the condition  $G = \mathrm{SO}(m, 1)$ . The conjecture of the same equidistribution result for general case is left as an open problem at the end of that paper. And it is also selected as an unsolved conjecture in Gorodnik's survey [1] (see [1, Conjecture 19]). In this article, we will answer this question positively by showing some observations on the representations of  $\mathrm{SL}(2, \mathbb{R})$ .

Here is the plan of this paper: In section 2, we recall some basic facts on the structure of  $\mathrm{SO}(n, 1)$  and the geodesic flows on  $T^1(\mathbb{H}^n) \cong \mathrm{SO}(n, 1)/\mathrm{SO}(n-1)$ ; in section 3, we follow the argument in [3] to show that the limit measure of the evolutions of the normalized measure on our original curve under the action of geodesic flow is also a probability measure on  $G/\Gamma$  and is invariant under some unipotent subgroup; in section 4, assuming that the limit measure is not the one induced by the Haar measure on  $G$ , we use the techniques developed in [3] to get an algebraic condition concerning some particular representation of  $\mathrm{SO}(n, 1)$  on  $V$  and the orbit of some point  $v \in V$  under the action of the image of the curve, say  $u(\varphi(I))v$ ; in section 5, we will prove a technical result (Lemma 5.1) concerning the representations of  $\mathrm{SL}(2, \mathbb{R})$  that allows us to simplify the condition we get in section 4; in section 6, we combine the condition we get in section 4 and Lemma 5.1 to complete the proofs of Theorem 2.1 and then Theorem 1.1.

## 2 Preliminaries

We realize  $H = \mathrm{SO}(n, 1)$  as a matrix group preserving the quadratic form  $Q$  in  $(n + 1)$  real variables defined as follows:

$$Q(x_0, x_1, \dots, x_n) = 2x_0x_n - (x_1^2 + \dots + x_{n-1}^2). \quad (3)$$

Then  $A = \left\{ a_t = \begin{bmatrix} e^t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & e^{-t} \end{bmatrix} \right\}$  is a maximal  $\mathbb{R}$ -split torus of  $H$ , and let  $A^+$  denote

$\{a_t\}_{t>0}$ . Then for any  $h \in H \cong T^1(\mathbb{H}^n)$ , the orbit of  $h$  under geodesic flow  $\{g_t h\}_{t>0}$  is just the orbit under left multiplications of  $A^+$ , say  $\{a_t h\}_{t>0}$ . If we define  $\alpha : A \rightarrow \mathbb{R}^+$  by  $\alpha(a_t) = e^{t/2}$ , then  $A^+ = \{a \in A : \alpha(a) > 1\}$ . Next, let  $K$  be a maximal compact subgroup of  $H$ , and let  $M = Z_H(A) \cap K$ , where  $Z_H(A)$  denotes the centralizer of  $A$  in  $H$ . Then elements in  $M$  have the following form:

$$m = \begin{bmatrix} 1 & \mathbf{0}^T & 0 \\ \mathbf{0} & k(m) & \mathbf{0} \\ 0 & \mathbf{0}^T & 1 \end{bmatrix} \quad (4)$$

Where  $k(m) \in \mathrm{SO}(n - 1)$  and  $k : M \rightarrow \mathrm{SO}(n - 1)$  is an isomorphism. We can easily deduce that  $Z_H(A) = MA$ . Define:

$$N^- = \{h \in H : a^k h a^{-k} \rightarrow e \text{ as } k \rightarrow \infty \text{ for any } a \in A^+\} \quad (5)$$

$$N = \{h \in H : a^{-k} h a^k \rightarrow e \text{ as } k \rightarrow \infty \text{ for any } a \in A^+\} \quad (6)$$

Then  $P^- = MAN^-$  is a minimal parabolic subgroup of  $H$ . Let  $\mathfrak{n}$  be the Lie algebra of  $N$ . We know that  $\mathfrak{n} \cong \mathbb{R}^{n-1}$ , and if we denote the exponential map by  $u : \mathfrak{n} \cong \mathbb{R}^{n-1} \rightarrow N$ , then for  $\mathbf{x} = (x_1, x_2, \dots, x_{n-1})^T$ ,  $u(\mathbf{x})$  can be explicitly written as follows:

$$u(\mathbf{x}) = \begin{bmatrix} 1 & x_1 & \dots & x_{n-1} & \|\mathbf{x}\|^2/2 \\ & 1 & & & x_1 \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1} \\ & & & & 1 \end{bmatrix}$$

Similarly, for  $\mathfrak{n}^-$ , the Lie algebra of  $N^-$ , we can define  $u^- : \mathbb{R}^{n-1} \rightarrow N^-$ . And moreover, for  $\mathbf{x} \neq \mathbf{0}$ , the one-parameter unipotent subgroup  $\{u(t\mathbf{x}) : t \in \mathbb{R}\}$ , one-parameter unipotent subgroup  $\{u^-(t\mathbf{x}) : t \in \mathbb{R}\}$  and the diagonal subgroup  $A$  can be embedded into a subgroup of  $H$  which is isomorphic to  $SL(2, \mathbb{R})$ . In

this subgroup,  $u(t\mathbf{x})$  corresponds to  $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ ,  $a_t$  corresponds to  $\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$  and  $u^-(t\mathbf{x})$  corresponds to  $\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$

The action of  $M \cong \mathrm{SO}(n-1)$  on  $\mathbb{R}^{n-1}$  is defined by  $u(z\mathbf{x}) = zu(\mathbf{x})z^{-1}$  for  $z \in M$ . We will first prove the following special case of the main theorem:

**Theorem 2.1.** *Let  $\varphi : I = [a, b] \rightarrow \mathbb{R}^{n-1}$  be an analytic curve which is not contained in any sphere or affine hyperplane. Let  $x_i \xrightarrow{i \rightarrow \infty} x$  as be a convergent sequence in  $G/\Gamma$  and let  $\{a_i\}_{i \in \mathbb{N}}$  be a sequence in  $A_+$  such that  $\alpha(a_i) \xrightarrow{i \rightarrow \infty} \infty$ . Then for and  $f \in C_c(G/\Gamma)$*

$$\lim_{i \rightarrow \infty} \frac{1}{|I|} \int_I f(a_i u(\varphi(s))x_i) ds = \int_{G/\Gamma} f d\mu_G. \quad (7)$$

This will be the main part of the article.

### 3 Limit of normalized measures on the curve under geodesic flow and relate it to a unipotent subgroup

From now on, we consider Theorem 2.1 at first. For simplicity, we assume  $\dot{\varphi}(s) \neq 0$  for any  $s \in I$ . Then by reparametrization, we may assume  $|\dot{\varphi}(s)| = 1$  for all  $s \in I$ . Then there exists a continuous function  $z : I \rightarrow M$  such such  $z(s)u(\dot{\varphi}(s))z^{-1}(s) = u(e_1)$ , where  $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n-1}$ . Next, let  $\{a_i\} \subset A^+$  be a sequence such that  $\alpha(a_i) \xrightarrow{i \rightarrow \infty} \infty$  and let  $x_i \xrightarrow{i \rightarrow \infty} x$ . We define  $\lambda_i$  and  $\mu_i$  as follows:

$$\int_{G/\Gamma} f d\lambda_i := \frac{1}{|I|} \int_I f(z(s)a_i u(\varphi(s))x_i) ds \quad (8)$$

$$\int_{G/\Gamma} f d\mu_i := \frac{1}{|I|} \int_I f(a_i u(\varphi(s))x_i) ds. \quad (9)$$

For any  $f \in C_c(G/\Gamma)$ .

In this section, we will prove the following result:

**Theorem 3.1.** *Given  $\epsilon > 0$ , there exists compact set  $K \subset G/\Gamma$  such that  $\mu_i(K) \geq 1 - \epsilon$ . The same is true for  $\lambda_i$ .*

The prove is borrowed from [3, Section 2].

Let  $\mathcal{H}$  denote the collection of analytic subgroups  $L$  of  $G$  such that  $L \cap \Gamma$  is a lattice of  $L$ , one can prove that  $\mathcal{H}$  is a countable set (see [8]).

Let  $V = \bigoplus_{d=1}^{\dim \mathfrak{g}} \wedge^d \mathfrak{g}$  and consider the linear action of  $g \in G$  on  $V$  via  $\bigoplus_{d=1}^{\dim \mathfrak{g}} \wedge^d \mathrm{Ad} g$ . For  $L \in \mathcal{H}$ , let  $p_H \in V$  be  $e_1 \wedge e_2 \wedge \dots \wedge e_k$ , where  $\{e_1, e_2, \dots, e_k\}$  is a basis of the Lie algebra  $\mathfrak{l}$  of  $L$ . The following proposition is proved in [2]:

**Proposition 3.1.** *For  $L \in \mathcal{H}$ , the set  $\Gamma p_L \subset V$  is discrete.*

Let  $\Upsilon : I \rightarrow \text{End}(V)$  be the map given by  $\Upsilon(s) = \bigwedge \text{Ad}(u(\varphi(s)))$ . Fix  $s_0 \in I$ , since  $\Upsilon$  is analytic, it is easy to see that  $\Upsilon(s) \in \Upsilon(s_0) + \mathcal{E}_s$ , where  $\mathcal{E}_s := \text{span}\{\Upsilon^{(k)}(s) : k \geq 1\}$ , here  $\Upsilon^{(k)}(s)$  denotes the  $k$ -th derivative at  $s$ , and that  $\mathcal{E}_s$  does not change for different  $s$  and is the smallest subspace  $\mathcal{E}$  of  $\text{End}(V)$  satisfying  $\Upsilon(I) \subset \Upsilon(s_0) + \mathcal{E}$ .

We denote by  $\mathcal{F}$  the linear span of coordinate functions of  $\Upsilon$ .

By [6, Proposition 3.4], applied to the function  $s \mapsto \Upsilon(s) - \Upsilon(s_0)$ , there exist constants  $C > 0$  and  $\alpha > 0$  such that, for any subinterval  $J \subset I$ ,  $\xi \in \mathcal{F}$ , and  $r > 0$ , we have:

$$|\{s \in J : |\xi(s)| < r\}| \leq C \left( \frac{r}{\sup_{s \in J} |\xi(s)|} \right)^\alpha |J|. \quad (10)$$

The functions satisfying the above property are called  $(C, \alpha)$ -good functions.

Let  $\mathcal{F}(G)$  be the collection of all functions  $\psi : I \rightarrow G$  such that for any  $p \in V$  and for any linear functional  $f$  on  $V$ , if we define  $\xi(s) = f(\psi(s)p)$  for all  $s \in I$ , then  $\xi \in \mathcal{F}$ .

We have the following proposition.

**Proposition 3.2.** *Fix a norm  $\|\cdot\|$  on  $V$ . There exist closed subgroups  $W_1, \dots, W_r$  in  $\mathcal{H}$  such that the following holds: for any  $\epsilon > 0$  and  $R > 0$ , there exists a compact set  $K \subset G/\Gamma$  such that for any  $\psi \in \mathcal{F}(G)$  and any subinterval  $J \subset I$ , one of the following is satisfied:*

(I) *There exist  $\gamma \in \Gamma$  and  $j \in \{1, \dots, r\}$  such that*

$$\sup_{s \in J} \|\psi(s)\gamma p_{W_j}\| < R$$

(II)  $|\{s \in J : \pi(\psi(s)) \in K\}| \geq (1 - \epsilon)|J|$

For the proof, the reader is referred to [6] and the remark made after [3, Proposition 2.2].

Consider a linear representation of  $\text{SL}(2, \mathbb{R})$  on a finite-dimensional vector space  $V$ . Let  $a = \begin{bmatrix} \alpha & \\ & \alpha^{-1} \end{bmatrix}$  for some  $\alpha > 1$ , and define

$$\begin{aligned} V^+ &= \{v \in V : a^{-k}v \xrightarrow{k \rightarrow \infty} 0\} \\ V^0 &= \{v \in V : av = v\} \\ V^- &= \{v \in V : a^k v \xrightarrow{k \rightarrow \infty} 0\} \end{aligned} \quad (11)$$

The any  $v \in V$  can be uniquely expressed as  $v = v^+ + v^0 + v^-$ , where  $v^\pm \in V^\pm$  and  $v^0 \in V^0$ . We write  $V^{+0} = V^+ + V^0$  and  $V^{0-} = V^0 + V^-$ . Let  $q^+ : V \rightarrow V^+$ ,  $q^0 : V \rightarrow V^0$ ,  $q^{+0} : V \rightarrow V^{+0}$  and  $q^{0-} : V \rightarrow V^{0-}$  denote the projections. Let  $v^{+0} = q^{+0}(v) = v^+ + v^0$ ,  $v^{0-} = q^{0-}(v) = v^0 + v^-$ . We consider the Euclidean norm on  $V$  such that  $V^+$ ,  $V^0$  and  $V^-$  are orthogonal.

The following lemma is crucial for proving the nondivergence of the limit measure.

**Lemma 3.1.** *See [3, Lemma 2.3] Let  $\mathbf{u} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$  for some  $t \neq 0$ . Then there exists a constant  $\kappa = \kappa(t) > 0$  such that*

$$\max\{\|v^+\|, \|(uv)^{+0}\|\} \geq \kappa\|v\|, \forall v \in V \quad (12)$$

Since we can embed  $A$  and  $\mathbf{u} \in N \setminus e$  into a subgroup of  $H$  which is isomorphic to  $SL(2, \mathbb{R})$ , we get the following practical corollary:

**Corollary 3.1.** *See [3, Corollary 2.4] Let  $V$  be a finite-dimensional normed linear space. Consider a linear representation of  $H = SO(n, 1)$  on  $V$ , where  $n \geq 2$ . Let*

$$\begin{aligned} V^+ &= \{v \in V : a^{-k}v \xrightarrow{k \rightarrow \infty} 0 \text{ for any } a \in A^+\} \\ V^0 &= \{v \in V : av = v \text{ for any } a \in A^+\} \\ V^- &= \{v \in V : a^k v \xrightarrow{k \rightarrow \infty} 0 \text{ for any } a \in A^+\} \end{aligned} \quad (13)$$

Then given a compact set  $F \subset N \setminus \{e\}$ , there exists a constant  $\kappa > 0$  such that for any  $\mathbf{u} \in F$  and any  $v \in V$ ,

$$\max\{\|v^+\|, \|(\mathbf{u}v)^{+0}\|\} \geq \kappa\|v\| \quad (14)$$

In particular, for any  $a \in A^+$ , any  $\mathbf{u} \in F$  and any  $v \in V$ ,

$$\max\{\|av\|, \|\mathbf{u}av\|\} \geq \kappa\|v\| \quad (15)$$

Combining the previous proposition and corollary, we can deduce Theorem 3.1 as follows:

Let  $t_1, t_2 \in I$ , be such that  $\mathbf{u} := u(\varphi(t_2) - \varphi(t_1))^{-1} \neq e$ . Then there exists  $\kappa > 0$  such that

$$\max\{\|a_i v\|, \|a_i \mathbf{u}v\|\} \geq \kappa\|v\| \quad (16)$$

for any  $i \in \mathbb{N}$  and any  $v \in V$ . Let  $g_i \rightarrow g$  such that  $\pi(g_i) = x_i$ . By Proposition 3.1,  $\Gamma p_{W_j}$  is discrete in  $V$  for all  $j \in \{1, \dots, r\}$ . Therefore,

$$R_1 := \inf\{\|u(\varphi(t_1))g_i \gamma p_{W_j}\| : \gamma \in \Gamma, j \in \{1, \dots, r\}\} > 0$$

For any  $\gamma \in \Gamma$ ,  $\sigma \in \Sigma$  and  $i \in \mathbb{N}$ , if we put  $v = u(\varphi(t_1))g_i \gamma p_{W_j}$  in (16), then have

$$\sup_{t \in \{t_1, t_2\}} \{\|a_i u(\varphi(t))g_i \gamma p_{W_j}\|\} \geq \kappa\|u(\varphi(t_1))g_i \gamma p_{W_j}\| \geq \kappa R_1 \quad (17)$$

Given that  $\epsilon > 0$ , we obtain a compact set  $K \in G/\Gamma$  such that the conclusion of Proposition 3.2 holds for  $R = (1/2)\kappa R_1$ . Then by (17), for any  $i \in \mathbb{N}$ , possibility (I) of Proposition 3.2 does not hold. Therefore, possibility (II) will be true. This proves Theorem 3.1

From Theorem 3.1, we can immediately get the following fact: after passing a subsequence, we have  $\mu_i \rightarrow \mu$  and  $\lambda_i \rightarrow \lambda$  in the weak\*-topology, as  $i \rightarrow \infty$ , where  $\mu$  and  $\lambda$  are both probability measure on  $G/\Gamma$ . That is,

$$\lim_{i \rightarrow \infty} \int_{G/\Gamma} f d\mu_i = \int_{G/\Gamma} f d\mu \quad (18)$$

$$\lim_{i \rightarrow \infty} \int_{G/\Gamma} f d\lambda_i = \int_{G/\Gamma} f d\lambda \quad (19)$$

For any  $f \in C_c(G/\Gamma)$ .

Define

$$W = \{u(te_1) : t \in \mathbb{R}\}$$

. We have the following theorem:

**Theorem 3.2.** *The measure  $\lambda$  is  $W$ -invariant.*

*Proof.* See [3, Theorem 3.1]. □

Here is vague explanation why this is true. In fact, for any  $r \in \mathbb{R}$ ,  $s \in I$  and  $f \in C_c(G/\Gamma)$ , and for  $i$  large enough, we have:

$$\begin{aligned} & f(u(re_1)z(s)a_i u(\varphi(s))x_i) \\ &= f(z(s)u(r\dot{\varphi}(s))a_i u(\varphi(s))x_i) \\ &= f(z(s)a_i u(re^{-t_i}\dot{\varphi}(s) + \varphi(s))x_i) \\ &\approx f(z(s)a_i u(\varphi(s + re^{-t_i}))x_i) \end{aligned} \tag{20}$$

So for  $i \gg 1$ , the translation of  $f$  under  $u(re_1)$  is very close to  $f$ , after the normalized integral on  $I$  and let  $i \rightarrow \infty$ , we have  $u(re_1)\lambda = \lambda$ .

Next we may apply Ratner's theorem.

For  $L \in \mathcal{H}$ , define

$$N(L, W) = \{g \in G : g^{-1}Wg \subset L\}$$

and the associated singular set is defined as follows:

$$S(L, W) = \bigcup_{\substack{F \in \mathcal{H} \\ F \subsetneq L}} N(F, W)$$

By [5, Proposition 2.1, Lemma 2.4],

$$N(L, W) \cap N(L, W)\gamma \subset S(L, W), \text{ for any } \gamma \in \Gamma \backslash N_G^1(L) \tag{21}$$

where  $N_G^1(L) = \{g \in N_G(L) : \det(\text{Ad}(g)|_{\mathfrak{l}}) = 1\}$ .

By Ratner's theorem, we have the following.

**Theorem 3.3.** (Ratner) *Given the  $W$ -invariant probability measure  $\lambda$  on  $G/\Gamma$ , there exists  $L \in \mathcal{H}$  such that*

$$\lambda(\pi(N(L, W))) > 0 \quad \text{and} \quad \lambda(\pi(S(L, W))) = 0 \tag{22}$$

*Moreover, almost every  $W$ -ergodic component of  $\lambda$  on  $\pi(N(L, W))$  is a measure of the form  $g\mu_L$  where  $g \in N(L, W) \backslash S(L, W)$ ,  $\mu_L$  is a finite  $L$ -invariant measure on  $\pi(L)$ , and  $g\mu_L(E) = \mu(g^{-1}E)$  for all Borel sets  $E \subset G/\Gamma$ . In particular, if  $L \triangleleft G$ , then  $\lambda$  is  $L$ -invariant.*

We first consider the case  $L = G$ , under which we have  $\lambda = \mu_G$ . We will prove  $\mu = \mu_G$ .

We use the notation  $\eta_1 \stackrel{\epsilon}{\approx} \eta_2$  to say  $|\eta_1 - \eta_2| < \epsilon$ .

Let  $f \in C_c(G/\Gamma)$  and  $\epsilon > 0$  be given. Then there exists  $\delta > 0$  such that if  $J$  is any subinterval of  $I$  of length less than  $\delta$ , then  $f(z(t_0)^{-1}z(t)y) \stackrel{\epsilon}{\approx} f(y)$  for any  $t_0, t \in J$  and any  $y \in G/\Gamma$ .

If we define  $\lambda_i^J$  as in (8) for  $J$  in place of  $I$ . Then passing a subsequence we have  $\lambda_i^J \xrightarrow{i \rightarrow \infty} \lambda^J$ , where  $\lambda^J$  is a probability measure on  $G/\Gamma$  invariant under  $W$ -action. Note that  $\lambda^J(\pi(N(L, W))) = 0$  whenever  $\lambda(\pi(N(L, W))) = 0$ . So if  $\lambda = \mu_G$  so is  $\lambda^J$ .

Fix any  $t_0 \in J$ , and let  $f_0(y) = f(z(t_0)^{-1}y)$ . Then

$$\begin{aligned}
 & \int_J f(a_i u(\varphi(t))x_i) dt \\
 & \stackrel{\epsilon|J|}{\approx} \int_J f(z(t_0)^{-1}z(t)a_i u(\varphi(t))x_i) dt \\
 & = |J| \int_{G/\Gamma} f_0(y) d\lambda_i^J(y) \\
 & \stackrel{\epsilon|J|}{\approx} \int_{G/\Gamma} f_0(y) d\mu_G \text{ when } i > i_J \text{ for some } i_J
 \end{aligned} \tag{23}$$

Then by dividing  $I$  into finitely many subintervals  $J$  of length less than  $\delta$ , we get

$$|I| \int_{G/\Gamma} f d\mu_i \stackrel{3\epsilon|I|}{\approx} \int_{G/\Gamma} f d\mu_G \tag{24}$$

So for any  $\epsilon > 0$ , we have

$$\int_{G/\Gamma} f d\mu_i \stackrel{3\epsilon}{\approx} \int_{G/\Gamma} f d\mu_G \tag{25}$$

given  $i$  large enough. Therefore,  $\mu_i \rightarrow \mu_G$  as  $i \rightarrow \infty$ .

## 4 Linearization and singular sets

In this section, we consider the case that  $L \subseteq G$ .

As defined above, let  $V = \bigoplus_{d=1}^{\dim \mathfrak{g}} \wedge^d \mathfrak{g}$  and  $G$  act on  $V$  via exterior product of adjoint action  $\bigoplus_{d=1}^{\dim \mathfrak{g}} \wedge^d \text{Ad}_g$ . Then let  $l = \dim L$  and  $p_L = \wedge^l \mathfrak{l} \setminus \{0\}$ , where  $\mathfrak{l}$  denotes the Lie algebra of  $L$ .

Define  $\Gamma_L = \Gamma \cap N_G^1(L)$ , it is observed that

$$\Gamma_L = \{\gamma \in \Gamma : \gamma p_L = \pm p_L\} \tag{26}$$

For proof, the readers may see [2, Lemma 3.1].

Define  $\eta : G \rightarrow V$  by  $\eta(g) = g p_L$ , and let  $\mathcal{A}$  denote the Zariski closure of  $\eta(N(L, W))$ . In [2], the following equality is proved (see [2, Proposition 3.2]):

$$N(L, W) = \eta^{-1}(\mathcal{A}) \tag{27}$$

For any compact subset  $\mathcal{D} \subset \mathcal{A}$ , the singular set of  $\mathcal{D}$  is defined as follows:

$$S(\mathcal{D}) = \{g \in N(L, W) : \eta(g\gamma) \in \mathcal{D} \text{ for some } \gamma \in \Gamma \setminus \Gamma_L\} \tag{28}$$

In [3], the following propositions were proved:

**Proposition 4.1.** (see [3, Proposition 4.5])  $S(\mathcal{D}) \subset S(L, W)$  and  $\pi(S(\mathcal{D}))$  is closed in  $G/\Gamma$ . Moreover, for any compact set  $\mathcal{K} \in G/\Gamma \setminus \pi(S(\mathcal{D}))$ , there exists some neighborhood  $\Phi$  of  $\mathcal{D}$  in  $V$  such that, for any  $g \in G$  and  $\gamma_1, \gamma_2 \in \Gamma$ , if  $\pi(g) \in \mathcal{K}$  and  $\eta(g\gamma_i) \in \Phi$ ,  $i = 1, 2$ , then  $\eta(\gamma_1) = \pm \eta(\gamma_2)$ .

**Proposition 4.2.** (see [3, Proposition 4.6]) Given a symmetric compact set  $C \subset \mathcal{A}$  and  $\epsilon > 0$ , there exists a symmetric compact set  $\mathcal{D} \subset \mathcal{A}$  containing  $C$  such that, given a symmetric neighborhood  $\Phi$  of  $\mathcal{D}$  in  $V$ , there exists a symmetric neighborhood  $\Psi$  of  $C$  in  $V$  contained in  $\Phi$  such that for any  $\psi \in \mathcal{F}(G)$ , for any  $v \in V$ , and for any interval  $J \subset I$ , one of the following holds:



(I)  $\psi(t)v \in \Phi$  for all  $t \in J$

(II)  $|\{t \in J : \psi(t)v \in \Psi\}| \leq \epsilon |\{t \in J : \psi(t)v \in \Phi\}|$

**Proposition 4.3.** (see [3, Proposition 4.7]) Suppose that  $\epsilon > 0$ , compact set  $\mathcal{K} \subset G/\Gamma$ , open symmetric subsets  $\Phi, \Psi \subset V$ , a countable subset  $\Sigma \subset V$ , a bounded interval  $J$ , and a continuous map  $\psi : J \rightarrow G$  satisfy the following two conditions:

(I) For any  $g \in G$  and  $v_1, v_2 \in \Sigma$ , if  $gv_1, gv_2 \in \overline{\Phi}$  and if  $\pi(g) \in \mathcal{K}$ , then  $v_1 = \pm v_2$ .

(II) For any  $v \in \Sigma$ , if we define  $E_v = \{s \in J : \psi(s)v \in \Psi\}$  and  $F_v = \{s \in J : \psi(s)v \in \Phi\}$ , then  $|J_1 \cap E_v| \leq \epsilon |J_1|$  for any connected component  $J_1$  of  $F_v$ .

Then we have

$$|\{t \in J : \psi(t)\Sigma \cap \Psi \neq \emptyset, \pi(\psi(t)) \in \mathcal{K}\}| \leq 2\epsilon |J|$$

.

As defined above, we can decompose  $V = V^+ \oplus V^0 \oplus V^-$  according to under the action of any  $a \in A^+$ , if the vector is expanded, remain the same, or is contracted.

Now we are ready to prove the following crucial proposition:

**Proposition 4.4.** (see [3, Proposition 4.8]) Let  $\lambda$  and  $L \subset G$  be as defined above, if  $L \subsetneq G$ , then we have there exists  $v \in g\Gamma p_L$  such that, for any  $s \in I = [a, b]$ ,

$$u(\varphi(s))v \in V^- + V^0 \tag{29}$$

For the proof we follow the proof in [3, Proposition 4.8].

*Proof.* According to the definition of  $\lambda$  and  $L$ , there exists a compact set  $C \subset N(L, W) \setminus S(L, W)$  such that  $\lambda(\pi(C)) > c_0 > 0$  where  $c_0$  is some constant. Then we can define a symmetric compact set  $C = C_{p_L} \cup -C_{p_L} \subset \mathcal{A}$ . Then for some given  $0 < \epsilon < c_0/2$ , we can apply the claim in Proposition 4.2 to get a symmetric compact set  $\mathcal{D} \subset \mathcal{A}$  satisfying the conclusion of the proposition. Then we apply Proposition 4.1, there exists a compact set  $\mathcal{K} \subset G/\Gamma \setminus \pi(S(\mathcal{D}))$  such that  $\pi(C)$  is contained in the interior of  $\mathcal{K}$ , and there exists some symmetric neighborhood  $\Phi$  of  $\mathcal{D}$  in  $V$  such that the last conclusion of Proposition 4.1 holds. Then we can find a symmetric neighborhood  $\Psi$  of  $C$  in  $V$  such that the conclusion of Proposition 4.2 holds.

Define:

$$\Omega = \{\pi(g) : \pi(g) \in \mathcal{K}, gp_L \in \Psi\} \subset G/\Gamma \tag{30}$$

Then  $\pi(C) \subset \Omega$ , so we can choose some large  $i_0$  such that  $\lambda_i(\Omega) > c_0$  for  $i \geq i_0$ .

We choose a sequence  $g_i \rightarrow g$  such that  $\pi(g_i) = x_i$ . Then if we fix some  $i \geq i_0$ , we have

$$|\{t \in I : \pi(a_i z(t)u(\varphi(t))g_i) \in \Omega\}| > c_0 |I| \tag{31}$$

We denote  $\psi(t) = a_i z(t)u(\varphi(t))g_i$ , then  $\psi \in \mathfrak{F}(G)$ . Let  $\Sigma = \{v \in \Gamma p_L : \psi(I)v \not\subset \Phi\}$ .

For  $v \in \Sigma$ ,  $I$ ,  $\Psi$  and  $\Phi$ , we may define  $E_v$  and  $F_v$  as in Proposition 4.3, then for any interval  $F_1$  of  $F_v$ ,  $\overline{\psi(F_1)} \not\subseteq \Phi$ , therefore by Proposition 4.2, we have  $|E_v \cap F_1| \leq \epsilon |F_1|$ . Combining this with the choice of  $\mathcal{K}$ , the conditions of Proposition 4.3 satisfy. If  $\Sigma = \Gamma p_L$ , then

$$|\{t \in I : \pi(\psi(t)) \in \mathcal{K}, \psi(t)\Gamma p_L \cap \Psi \neq \emptyset\}| \leq 2\epsilon |I| \quad (32)$$

This is a contradiction of 31 since  $2\epsilon < \lambda$ . Therefore, for any  $i \geq i_0$ , there exists some  $\gamma_i \in \Gamma$ , such that

$$a_i z(t) u(\varphi(t)) g_i \gamma_i p_L \in \Phi, \forall t \in I \quad (33)$$

Since  $z(I)$  is compact, we have  $z(I)^{-1}\Phi$  is bounded. Suppose  $z(I)^{-1}\Phi \subset B(R)$ , where  $B(R)$  is a ball of radius  $R$  centered at zero. So

$$\|a_i u(\varphi(t)) g_i \gamma_i p_L\| \leq R, \forall t \in I \quad (34)$$

Fix  $t_0 \in I$ . Applying Corollary 3.1, we have there exists some  $\kappa > 0$  such that

$$\sup_{t \in I} \|q^{+0}(u(\varphi(t))v)\| \geq \kappa \|u(\varphi(t_0))v\|, \forall v \in V \quad (35)$$

Then we have  $\kappa \|u(\varphi(t_0)) g_i \gamma_i p_L\| \leq \sup_{t \in I} \|q^{+0}(u(\varphi(t)) g_i \gamma_i p_L)\| \leq \sup_{t \in I} \|a_i u(\varphi(t)) g_i \gamma_i p_L\| \leq R$ , therefore  $\|g_i \gamma_i p_L\| \leq \kappa^{-1} \|u(\varphi(t_0))^{-1}\| R$ . Since  $\Gamma p_L$  is discrete and  $g_i \rightarrow g$ , from the foregoing inequality we have  $\{\gamma_i p_L\}$  is finite. Then passing to a subsequence we can assume  $\gamma_i p_L = \gamma p_L$  for all  $i$ . Then we have

$$a_i z(t) u(\varphi(t)) g_i \gamma p_L \in \Phi, \forall t \in I, \forall i \in \mathbb{N} \quad (36)$$

If we consider  $w_i(t) := q^+(z(t) u(\varphi(t)) g_i \gamma p_L)$ , then we have  $\limsup_{i \rightarrow \infty} \|a_i w_i(t)\| < \infty$  for all  $t \in I$ . But since  $\alpha(a_i) \rightarrow \infty$  as  $i \rightarrow \infty$ , we have  $\|w_i(t)\| \rightarrow 0$  as  $i \rightarrow \infty$ . Therefore  $q^+(z(t) u(\varphi(t)) g \gamma p_L) = \lim_{i \rightarrow \infty} q^+(z(t) u(\varphi(t)) g_i \gamma p_L) = 0$ . This shows  $z(t) u(\varphi(t)) g \gamma p_L \in V^0 + V^-$ , and thus  $u(\varphi(t)) g \gamma p_L \in V^0 + V^-$  since  $z(t)$  preserves the weight space.

This completes the proof.  $\square$

## 5 Basic lemma on representation

In this section, we will prove the following basic lemma about representation of  $\mathrm{SL}(2, \mathbb{R})$

**Lemma 5.1.** *Let  $\rho : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathrm{GL}(V)$  be a real linear representation of  $\mathrm{SL}(2, \mathbb{R})$  on a vector space  $V$ . Let*

$$A = \{a_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} : t \in \mathbb{R}\} \quad A^+ = \{a_t\}_{t>0} \quad (37)$$

$$N = \{u_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{R}\} \quad (38)$$

$$N^- = \{u_t = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} : t \in \mathbb{R}\} \quad (39)$$

Then for  $a \in A^+$  we define

$$\begin{aligned} V^+ &= \{v \in V : a^{-k}v \xrightarrow{k \rightarrow \infty} 0\} \\ V^0 &= \{v \in V : av = v\} \\ V^- &= \{v \in V : a^k v \xrightarrow{k \rightarrow \infty} 0\} \end{aligned} \quad (40)$$

Let  $q^+ : V \rightarrow V^+$  denote the natural projection of  $V$  onto  $V^+$ , and  $q^0, q^-$  denote the projection onto  $V^0$  and  $V^-$  respectively. Then if for some  $v \in V^- + V^0$  we further have  $\rho(u_r)v \in V^- + V^0$ , then  $(\rho(u_r)v)^0 = \rho(\mathfrak{V})v^0$ , where  $\mathfrak{V} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  and  $v^0$  is the image of the foregoing mentioned projection  $q^0 : V \rightarrow V^0$ . Moreover, if  $v \in V^- \setminus \{0\}$ , then  $q^+(\rho(u_r)v) \neq 0$

*Proof.* At first we decompose  $V$  into direct sum of irreducible subspaces of  $\mathrm{SL}(2, \mathbb{R})$ , say  $V = \bigoplus_{i=1}^m V_i$ . Then for each irreducible component  $V_i$ , we have the corresponding projection  $q_i : V \rightarrow V_i$ . Since every irreducible representation of  $\mathrm{SL}(2, \mathbb{R})$  can be written as  $\mathrm{Span}_{\mathbb{R}}\{w_0, w_1, \dots, w_l\}$ , and for  $w_k$ , if we denote  $\mathfrak{n} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $\mathfrak{h} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ , then  $d\rho(\mathfrak{h})w_k = (l - 2k)w_k$  and  $d\rho(\mathfrak{n})w_k = kw_{k-1}$ , then  $\rho(u_r)w_k = \sum_{j=0}^k \binom{k}{j} r^{k-j} w_j$ . If  $l$  is odd, then  $V_i$  does not have contribution to  $V^0$ , so we just consider the case when  $l = 2p$  is even. Let  $v_i$  be the image of  $v$  under the projection  $q_i : V \rightarrow V_i$ , then since  $v \in V^- + V^0$  and  $\rho(u_r)v \in V^- + V^0$ , for  $v_i$  we also have  $v_i \in V^- + V^0$  and  $\rho(u_r)v_i \in V^- + V^0$ . Then we can assume that  $v_i = \sum_{k=p}^{2p} c_k w_k$  ( $w_p \in V^0, w_k \in V^-$  for  $k > p$ ). Next we calculate the  $w_p$  coefficient of  $\rho(u_r)v_i$ . In fact, according to the foregoing description of the representation, for  $j \leq p$ , the  $w_j$  coefficient of  $\rho(u_r)v_i$  is  $\sum_{k=p}^{2p} \binom{k}{j} r^{k-j} c_k$ . Then since  $\rho(u_r)v_i \in V^- + V^0$ , we have for  $j > p$ , its  $w_j$  coefficient, which is  $\sum_{k=p}^{2p} \binom{k}{j} r^{k-j} c_k$ , equals 0. Thus we get a series of equations:

$$\begin{aligned} \sum_{k=p}^{2p} \binom{k}{p-1} r^{k-p+1} c_k &= 0 \\ \sum_{k=p}^{2p} \binom{k}{p-2} r^{k-p+2} c_k &= 0 \\ &\vdots \\ \sum_{k=p}^{2p} \binom{k}{0} r^k c_k &= 0 \end{aligned} \quad (41)$$

From these equations we want to find out its  $w_p$  coefficient, say  $\sum_{k=p}^{2p} \binom{k}{p} r^{k-p} c_k$ .

Consider the following polynomial

$$f(x) = \sum_{k=p}^{2p} \binom{k}{p} r^{k-p} c_k x^{k-p} \quad (42)$$

Then the  $w_p$  coefficient is just  $f(1)$ . Next we define the operator  $\mathfrak{I}$  on polynomials by  $\mathfrak{I}g(x) = \int_0^x g(t)dt$ . Then we denote  $F_i = \mathfrak{I}^i f$ , then we have

$$F_i(x) = \sum_{k=p}^{2p} \binom{k}{p} r^{k-p} c_k \frac{1}{(k-p+1)(k-p+2)\dots(k-p+i)} x^{k-p+i} \quad (43)$$

Note the following equality about binomial coefficient

$$\binom{k}{p-i} = \binom{k}{p} \frac{p(p-1)\dots(p-i+1)}{(k-p+1)(k-p+2)\dots(k-p+i)}$$

, we have

$$F_i(x) = \frac{1}{r^i p(p-1)\dots(p-i+1)} \sum_{k=p}^{2p} \binom{k}{p-i} r^{k-p+i} c_k x^{k-p+i} \quad (44)$$

Then from the foregoing equations, we have  $F_i(1) = 0$ , for any  $i = 1, 2, \dots, p$ .

Next, for  $F_p(x)$ , since  $F_i(x) = F_p^{(p-i)}(x)$  ( $g^{(m)}$  stands for the  $m$ -th derivative of  $g$ ), then  $F_p^{(i)}(1) = 0$  for  $i = 0, 1, \dots, p-1$ . Moreover, since  $F_i(0) = 0$  from the definition, we have  $F_p^{(i)}(0) = 0$  for  $i = 0, 1, \dots, p-1$ . Then  $F_p(x)$  has  $x$  factor of multiplicity at least  $p$  and has  $x-1$  factor of multiplicity at least  $p$ . And apparently from the definition  $\deg F_p = 2p$ , then we have  $F_p(x) = Cx^p(x-1)^p$ . Then  $f(1) = F_p^{(p)}(1) = p!C$ . On the other hand,  $(-1)^p C$  is the  $x^p$  coefficient of  $F_p(x)$ , and this coefficient comes from the constant term of  $f(x)$ , which is  $c_p$ , divided by  $p!$ . Thus,  $(-1)^p C = \frac{c_p}{p!}$ . It follows that  $f(1) = (-1)^p c_p$ . In other words, the  $w_p$  coefficient of  $\rho(u_r)v_i$  is equal to the  $w_p$  coefficient of  $v_i$  multiplied by  $(-1)^p$ . Next note that, in the irreducible representation,  $\rho(\mathfrak{I})w_k = (-1)^k w_{2p-k}$ , in particular,  $\rho(\mathfrak{I})w_p = (-1)^p w_p$ . This shows that  $(\rho(u_r)v_i)^0 = \rho(\mathfrak{I})v_i^0$ . Keeping this fact in mind, we do the foregoing argument for every irreducible component, we can conclude that  $(\rho(u_r)v)^0 = \rho(\mathfrak{I})v^0$ .

For the second claim of the lemma, if we have  $v \in V^-$  and  $\rho(u_r)v \in V^- + V^0$ , then in the previous paragraph, we have  $C = 0$ , and thus  $F_p(x) \equiv 0$  which implies  $f(x) \equiv 0$ . This proves  $c_k = 0$  for all  $k = p, p+1, \dots, 2p$ , i.e.,  $v = \mathbf{0}$ . Thus the second part of the lemma is proved.

This completes the proof.  $\square$

## 6 Conclusion

Now we are ready to prove Theorem 2.1

*Proof of Theorem 2.1.* From Proposition 4.4, if  $L \neq G$ , then there exists  $v \in g\Gamma p_L$  such that  $u(\varphi(s))v \in V^- + V^0$  for all  $s \in I$ . In  $\text{SO}(n, 1)$ , we fix a particular  $\text{SL}(2, \mathbb{R})$ -copy generated by  $u(re_1)$ ,  $u^-(re_1)$  and  $A$ .

At first, we claim that  $(z(s)u(\varphi(s))v)^0$  is fixed by  $\text{SL}(2, \mathbb{R})$ . In fact, since  $u(\varphi(s))v \in V^- + V^0$  for any  $s$ , by taking derivative, we have  $d\rho(\dot{\varphi}(s))u(\varphi(s))v \in V^- + V^0$  (here we interpret  $\mathbb{R}^{n-1}$  as Lie algebra of  $N$ , as we mentioned before), by multiplying  $z(s)$  on the left, and since  $Z(A)$  preserves the weight spaces of  $A$ , we have  $z(s)d\rho(\dot{\varphi}(s))u(\varphi(s))v \in V^- + V^0$ . According to the definition of  $z(s)$ , we have  $d\rho(e_1)z(s)u(\varphi(s))v \in V^- + V^0$ . Therefore,  $d\rho(e_1)(z(s)u(\varphi(s))v)^0 = 0$  since otherwise it will be increased

to  $V^0$ . This proves that  $(z(s)u(\varphi(s))v)^0$  is fixed by  $u(re_1)$ . And since it is also fixed by  $A$ , we have it is fixed by this particular  $\mathrm{SL}(2, \mathbb{R})$ . This proves the claim.

For some fixed  $s_0 \in I$ , if we put  $v_0 = z(s_0)u(\varphi(s_0))v$ , then we have that  $v_0 \in V^- + V^0$  and  $(v_0)^0$  is fixed by the particular  $\mathrm{SL}(2, \mathbb{R})$ . Moreover, the stabilizer of  $(v_0)^0$  has the structure of  $\mathrm{SO}(k, 1) \times C$ , where  $C$  is a compact subgroup, in particular, for the unipotent elements contained in it, we have the following:

**claim 1.** *The set  $S = \{\mathbf{x} \in \mathbb{R}^{n-1} : u(\mathbf{x})(v_0)^0 = (v_0)^0\}$  is a proper subspace, in other words, there exists some vector  $\mathbf{w} \in \mathbb{R}^{n-1}$  such that if  $\mathbf{x} \in S$ , then  $\langle \mathbf{x}, \mathbf{w} \rangle = 0$ , where  $\langle, \rangle$  denotes the standard inner product.*

In fact, it is easy to see that  $S$  is a subspace of  $\mathbb{R}^{n-1}$  since  $u(\mathbf{x}_1 + \mathbf{x}_2) = u(\mathbf{x}_1)u(\mathbf{x}_2)$ . If  $S = \mathbb{R}^{n-1}$ , then  $u(\mathbf{x})(v_0)^0 = (v_0)^0$  for all  $\mathbf{x} \in \mathbb{R}^{n-1}$ . By embedding  $u(\mathbf{x})$  into an  $\mathrm{SL}(2, \mathbb{R})$  copy of  $\mathrm{SO}(n, 1)$  with  $A$  as the diagonal subgroup, we have  $(v_0)^0$  is invariant under the action of  $\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$  for all  $t \in \mathbb{R}$  and  $\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$  for

all  $r \in \mathbb{R}$ . From basic result of  $\mathrm{SL}(2, \mathbb{R})$  representation,  $(v_0)^0$  is also invariant under  $\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$  for all  $r \in \mathbb{R}$ .

This implies that  $u^-(\mathbf{x})$  also fixes  $(v_0)^0$  for all  $\mathbf{x} \in \mathbb{R}^{n-1}$ . Then the stabilizer of  $(v_0)^0$  must be the whole  $\mathrm{SO}(n, 1)$ . Let  $v^- = v_0 - (v_0)^0$ , then  $v^- \in V^-$ , and  $u(z(s_0)(\varphi(s) - \varphi(s_0)))v^- = u(\varphi(s))v - u(z(s_0)(\varphi(s) - \varphi(s_0)))(v_0)^0 = u(\varphi(s))v - (v_0)^0 \in V^- + V^0$  for  $s \in I$ . By embedding  $u(z(s_0)(\varphi(s) - \varphi(s_0)))$  into an  $\mathrm{SL}(2, \mathbb{R})$  copy of  $\mathrm{SO}(n, 1)$  with  $A$  as the diagonal subgroup, we can apply Lemma 5.1 to claim that  $v^- = \mathbf{0}$ . Therefore,  $v_0 = (v_0)^0$  is fixed by the whole  $\mathrm{SO}(n, 1)$  action, and so  $g\gamma p_L = (z(s_0)u(\varphi(s_0)))^{-1}v_0$  is fixed by  $\mathrm{SO}(n, 1)$  action since  $z(s_0)u(\varphi(s_0)) \in \mathrm{SO}(n, 1)$ . Then  $p_L$  is fixed by the action of  $\gamma^{-1}g^{-1}\mathrm{SO}(n, 1)g\gamma$ . Thus

$$\begin{aligned} \Lambda p_L &= \overline{\Lambda p_L} \text{ since } \Gamma p_L \text{ is discrete} \\ &= \overline{\Gamma \gamma^{-1} g^{-1} \mathrm{SO}(n, 1) g \gamma p_L} \\ &= \overline{\Gamma g^{-1} \mathrm{SO}(n, 1) g \gamma p_L} \\ &= G g \gamma p_L \text{ since in the condition } \overline{\mathrm{SO}(n, 1) g \Gamma} = G \\ &= G p_L \end{aligned} \tag{45}$$

This implies  $G_0 p_L = p_L$  where  $G_0$  is the connected component of  $e$ , in particular,  $\gamma^{-1}g^{-1}\mathrm{SO}(n, 1)g\gamma \subset G_0$  and  $G_0 \subset N_G^1(L)$ . By [4, Theorem 2.3], there exists a closed subgroup  $H_1 \subset N_G^1(L)$  containing all Ad-unipotent one-parameter subgroups of  $G$  contained in  $N_G^1(L)$  such that  $H_1 \cap \Gamma$  is a lattice in  $H_1$  and  $\pi(H_1)$  is closed. If we put  $F = g\gamma H_1 \gamma^{-1} g^{-1}$ , then  $\mathrm{SO}(n, 1) \subset F$  since  $\mathrm{SO}(n, 1)$  is generated by it unipotent one-parameter subgroups. Moreover,  $Fx = g\gamma\pi(H_1)$  is closed and admits a finite  $F$ -invariant measure. Then since  $\overline{\mathrm{SO}(n, 1)x} = G/\Gamma$ , we have  $F = G$ . This implies  $H_1 = G$  and thus  $L \triangleleft G$ . Therefore  $N(L, W) = G$ . In particular,  $W \subset L$ , and thus  $L \cap \mathrm{SO}(n, 1)$  is a normal subgroup of  $\mathrm{SO}(n, 1)$  containing  $W$ . Since  $\mathrm{SO}(n, 1)$  is a simple group, we have  $\mathrm{SO}(n, 1) \subset L$ . Since  $L$  is a normal subgroup of  $G$  and  $\pi(L)$  is a closed orbit with finite  $L$ -invariant measure, every orbit of  $L$  on  $G/\Gamma$  is also closed and admits a finite  $L$ -invariant measure, in particular,  $Hx$  is closed. But since  $\mathrm{SO}(n, 1)x$  is dense in  $G/\Gamma$ ,  $Lx$  is also dense. This shows that  $L = G$ , which contradicts to our hypothesis. This proves the claim. Then for

$v = gyp_L$ , the set  $T = \{\mathbf{x} \in \mathbb{R}^{n-1} : u(\mathbf{x})v^0 = v^0\}$  is also a proper subspace of  $\mathbb{R}^{n-1}$ , i.e., there exists  $\mathbf{v} \in \mathbb{R}^{n-1}$ , such that for any  $\mathbf{x} \in \mathbb{R}^{n-1}$ , we have  $\langle \mathbf{x}, \mathbf{v} \rangle = 0$ .

Assume  $\varphi(s) = r(s)k(s)e_1$ , where  $r(s) \in \mathbb{R}$  and  $k(s) \in \text{SO}(n-1) \cong M$ . For simplicity, for  $k \in \text{SO}(n-1)$ , we still use  $k$  to denote its corresponding element in  $M$ . So we have  $u(k\mathbf{x}) = ku(\mathbf{x})k^{-1}$  for any  $k \in \text{SO}(n-1)$  and  $\mathbf{x} \in \mathbb{R}^{n-1}$ .

Then

$$\begin{aligned} u(\varphi(s))v &= u(r(s)k(s)e_1)v \\ &= k(s)u(r(s)e_1)k^{-1}(s)v \end{aligned} \quad (46)$$

Since  $k(s)$  preserves the weight spaces of  $A$ , we have  $k^{-1}(s)v \in V^- + V^0$  and  $u(r(s)e_1)k^{-1}(s)v$ . Applying Lemma (5.1) with the prescribed  $\text{SL}(2, \mathbb{R})$  (generated by  $u(re_1)$ ,  $A$  and  $u^-(re_1)$ ) and  $v = k^{-1}(s)v$ , we have  $(u(r(s)e_1)k^{-1}(s)v)^0 = \mathfrak{J}(k^{-1}(s)v_0)^0 = \mathfrak{J}k^{-1}(s)v_0^0$ , where  $\mathfrak{J}$  corresponds to  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  in the prescribed

$\text{SL}(2, \mathbb{R})$ , which corresponds to  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & E & 0 \\ 1 & 0 & 0 \end{bmatrix}$  where  $E = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ . Thus

$$\begin{aligned} (z(s)u(\varphi(s))v)^0 &= z(s)(u(\varphi(s))v)^0 \\ &= z(s)(k(s)u(r(s)e_1)k^{-1}(s)v)^0 \\ &= z(s)k(s)(u(re_1)k^{-1}(s)v)^0 \\ &= z(s)k(s)\mathfrak{J}k^{-1}(s)v^0 \end{aligned} \quad (47)$$

From the preceding claim,  $(z(s)u(\varphi(s))v)^0 = z(s)k(s)\mathfrak{J}k^{-1}(s)v^0$  is fixed by the particular  $\text{SL}(2, \mathbb{R})$ . Let  $h$  denote  $z(s)k(s)\mathfrak{J}k^{-1}(s)$ , then we have  $v$  is fixed by  $h^{-1}\text{SL}(2, \mathbb{R})h$ . In particular, it is fixed by  $h^{-1}u^-(e_1)h$ . By calculation,

$$\begin{aligned} h^{-1}u^-(e_1)h &= k(s)\mathfrak{J}k^{-1}(s)z^{-1}(s)u^-(e_1)z(s)k(s)\mathfrak{J}k^{-1}(s) \\ &= u(k(s)Ek^{-1}(s)z^{-1}(s)e_1) \end{aligned} \quad (48)$$

From the foregoing argument, we have  $\langle k(s)Ek^{-1}(s)z^{-1}(s)e_1, \mathbf{v} \rangle = 0$ . According to the definition of  $z(s)$ ,  $z^{-1}(s)e_1 = \dot{\varphi}(s)$ , thus,  $\langle k(s)Ek^{-1}(s)\dot{\varphi}(s), \mathbf{v} \rangle = 0$ .

From  $\varphi(s) = r(s)k(s)e_1$ , we have  $\dot{\varphi}(s) = \dot{r}(s)k(s)e_1 + r(s)\dot{k}(s)e_1$ , so

$$\begin{aligned} k(s)Ek^{-1}(s)\dot{\varphi}(s) &= k(s)Ek^{-1}(s)(\dot{r}(s)k(s)e_1 + r(s)\dot{k}(s)e_1) \\ &= k(s)Ek^{-1}(s)\dot{r}(s)k(s)e_1 + k(s)Ek^{-1}(s)r(s)\dot{k}(s)e_1 \\ &= \dot{r}(s)k(s)Ee_1 + r(s)k(s)Ek^{-1}(s)\dot{k}(s)e_1 \\ &= -\dot{r}(s)k(s)e_1 + r(s)k(s)Ek^{-1}(s)\dot{k}(s)e_1 \end{aligned} \quad (49)$$

For  $r(s)k(s)Ek^{-1}(s)\dot{k}(s)e_1$ , we put  $k(s) = [a_1, \dots, a_{n-1}]$ , where  $a_i \in \mathbb{R}^{n-1}$  are column vectors, it is clearly that  $k^{-1}(s) = k(s)^t$  since  $k(s) \in \text{SO}(n-1)$ . Then  $\dot{k}(s) = [\dot{a}_1, \dots, \dot{a}_{n-1}]$ , so  $\dot{k}(s)e_1 = \dot{a}_1$ . Then, the first coordinate of  $k^{-1}(s)\dot{k}(s)e_1 = a_1^t \dot{a}_1 = \langle a_1, \dot{a}_1 \rangle$ . Given  $k(s) \in \text{SO}(n-1)$ , we have  $\langle a_1, a_1 \rangle = 1$ ,

by taking derivative, we get  $\langle a_1, \dot{a}_1 \rangle = 0$ , i.e., the first coordinate of  $k^{-1}(s)\dot{k}(s)e_1$  is zero. This follows that it is fixed by  $E$ , thus  $k(s)Ek^{-1}(s)\dot{k}(s)e_1 = k(s)k^{-1}(s)\dot{k}(s)e_1 = \dot{k}(s)e_1$ . Combining them, we have

$$\begin{aligned} k(s)Ek^{-1}(s)\dot{\varphi}(s) &= -\dot{r}(s)k(s)e_1 + r(s)\dot{k}(s)e_1 \\ &= \dot{\varphi}(s) - 2\dot{r}(s)k(s)e_1 \\ &= \dot{\varphi}(s) - 2\dot{r}(s)\varphi(s)/r(s) \end{aligned} \quad (50)$$

Therefore,  $\langle \dot{\varphi}(s) - 2\dot{r}(s)\varphi(s)/r(s), \mathbf{v} \rangle = 0$  for any  $s \in I$ , this means  $\frac{d}{ds}(\frac{\langle \varphi(s), \mathbf{v} \rangle}{r^2(s)}) = 0$ , which implies  $\frac{\langle \varphi(s), \mathbf{v} \rangle}{r^2(s)} = C$  is a constant. For  $C = 0$ , this gives an equation of hyperplane, for  $C \neq 0$ , this gives an equation for a subsphere.

This completes the proof.  $\square$

Finally, we explain how to get Theorem 1.1 from Theorem 2.1.

*Proof of Theorem 1.1.* In fact, the geodesic flow  $g_t$  just corresponds to the action of  $a_t$  on the left if we identify  $H$  with  $T^1(\mathbb{H}^n)$ . Let  $\phi : I \rightarrow H := \text{SO}(n, 1)$  be any analytical map, then there exists some  $\varphi : I \rightarrow \mathbb{R}^{n-1}$  such that  $\phi(t) = N^-(t)K(t)u(\varphi(t))$  where  $N^-(t) \in N^-$ , and  $K(t) \in AM$ . Then  $\text{Vis}(\phi(I))$  is contained in a proper subsphere if and only if  $\varphi(I)$  is contained in a proper subsphere or hyperplane. Then for any  $f \in C_c(G/\Gamma)$ , if we consider the normalized line integral

$$\begin{aligned} &\frac{1}{|I|} \int_I f(g_t \phi(s)x) ds \\ &= \frac{1}{|I|} \int_I f(a_t N^-(s)K(s)u(\varphi(s))x) ds \\ &= \frac{1}{|I|} \int_I f(a_t N^-(s)a_t^{-1}K(s)a_t u(\varphi(s))x) ds \end{aligned} \quad (51)$$

Since  $N(I) \subset N^-$  compact,  $f \in C_c(G/\Gamma)$ , and  $a_t N^-(I)a_t^{-1} \rightarrow e$  as  $t \rightarrow +\infty$ , we have for any given  $\epsilon > 0$ , there exists some  $t_0$ , such that for  $t > t_0$  large enough,  $f(a_t N(s)a_t^{-1}K(s)a_t u(\varphi(s))x) \approx^\epsilon f(K(s)a_t u(\varphi(s))x)$  for all  $s \in I$ . This means  $\int_I f(g_t \phi(s)x) ds \approx^{\epsilon|I|} \int_I f(K(s)a_t u(\varphi(s))x) ds$ .

Next, from the previous assumption,  $\varphi(I)$  is not contained in any proper subsphere or hyperplane, and then the same is also true for any subinterval  $J \subset I$ , due to the analyticity of  $\varphi$ . Then  $\frac{1}{|J|} \int_J f(a_t u(\varphi(s))x) ds \rightarrow \int_{G/\Gamma} f(x) d\mu_G$  as  $t \rightarrow +\infty$  for any subinterval  $J$ . Since  $f \in C_c(G/\Gamma)$ , we can divide  $I$  into several subintervals  $J_1, \dots, J_k$  such that  $f(K(s_0)K(s)^{-1}x) \approx^\epsilon f(x)$  for any  $s_0, s$  in the same subinterval. Since for each subinterval the normalized line integral tends to  $\int_{G/\Gamma} f(x) d\mu_G$ , we can choose some  $t_1$  large enough such that  $\int_{J_i} f(a_t u(\varphi(s))x) \approx^{\epsilon|J_i|} \int_{G/\Gamma} f d\mu_G$  for all  $J_i$  and  $t > t_1$ . For some  $J_i$ , we fix a point  $s_0 \in J_i$  and denote  $f_0(x) = f(K(s_0)x)$ . Then for  $t > t_1$ ,

$$\begin{aligned} \int_{J_i} f(K(s)a_t u(\varphi(s))x) ds &\stackrel{\epsilon|J_i|}{\approx} \int_{J_i} f(K(s_0)a_t u(\varphi(s))x) ds \\ &= \int_{J_i} f_0(a_t u(\varphi(s))x) ds \\ &\stackrel{\epsilon|J_i|}{\approx} |J_i| \int_{G/\Gamma} f_0 d\mu_G \\ &= \int_{G/\Gamma} f d\mu_G \end{aligned} \quad (52)$$

The last equality holds since  $\mu_G$  is left invariant. Then by adding all integrals over  $J_i$ , we get

$$\int_I f(K(s)a_t u(\varphi(s))x) ds \approx^{2\epsilon|I|} |I| \int_{G/\Gamma} f d\mu_G \quad (53)$$

i.e.,

$$\frac{1}{|I|} \int_I f(K(s)a_t u(\varphi(s))x) ds \stackrel{2\epsilon}{\approx} \int_{G/\Gamma} f d\mu_G \quad (54)$$

Combining the fact that  $f(g_t \phi(s)x) \stackrel{\epsilon}{\approx} f(K(s)a_t u(\varphi(s))x)$  for  $t > t_0$ , we have for  $t > \max\{t_0, t_1\}$ ,

$$\frac{1}{|I|} \int_I f(g_t \phi(s)x) ds \stackrel{3\epsilon}{\approx} \int_{G/\Gamma} f d\mu_G \quad (55)$$

Since  $\epsilon > 0$  is chosen arbitrarily, we have Theorem 1.1 proved.  $\square$

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